

Cyclotomic Temperley-Lieb algebra of type D and its representation theory

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Abstract

We define a new class of algebras, cyclotomic Temperley-Lieb algebras of type D , in a diagrammatic way, which is a generalization of Temperley-Lieb algebras of type D . We prove that the cyclotomic Temperley-Lieb algebras of type D are cellular. In fact, an explicit cellular basis is given by means of combinatorial methods. After determining all the irreducible representations of these algebras, we give a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebra of type D to be quasi-hereditary.

Key Words: cyclotomic Temperley-Lieb algebras of type D , cellular algebras, irreducible representations, quasi-hereditary algebras.

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1 Introduction

Temperley-Lieb algebras are a class of finite dimensional associative algebras first introduced by Temperley and Lieb (1971) in their analysis of Potts models and later rediscovered by Jones (1983) to characterize his algebras arising from the tower construction of semi-simple algebras in the study of subfactors. As well as having many applications to physics, Temperley-Lieb algebras are also of great value in several areas of mathematics, including the theory of quantum groups and knot theory, where they are closely related to the Jones polynomial and isotopy invariants of links. This relationship was explained in Jones (1987), where it was shown that Temperley-Lieb algebras occur naturally as quotients of Hecke algebras arising from a Coxeter system of type A . In his Ph.D. thesis, Graham (1995) studied certain quotients of Hecke algebras associated to a Coxeter diagram X , which were called Temperley-Lieb algebras of type X . Graham classified finite dimensional Temperley-Lieb algebras into seven infinite families: A , B , D , E , F , H and I . Some affine versions of Temperley-Lieb

algebras have also been studied by Graham and Lehrer (1998), Fan and Green (1999), and so on.

Recently, cyclotomic Temperley-Lieb algebras of type A , as a generalization of Temperley-Lieb algebras of type A , were introduced and investigated by Rui and Xi (2004) and were proved to be cellular in the sense of Graham and Lehrer (1996) by means of dotted planar graphs. Using the theory of cellular algebras (König and Xi, 1998, 1999a, 1999c) all irreducible representations of cyclotomic Temperley-Lieb algebras of type A were parametrised by Rui and Xi (2004). They also determined when these algebras are quasi-hereditary (in the sense of Cline, Parshall and Scott, 1988). Moreover, a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebra of type A to be semi-simple was provided in Rui, Xi and Yu (2005). Martin and Saleur (1994) introduced blob algebras which can be seen as another generalization of the Temperley-Lieb algebras of type A by adding an idempotent generator and some defining relations. Cao and Zhu (2006) defined cyclotomic blob algebras as algebras of diagrams and showed that cyclotomic blob algebras are cellular.

A common approach when studying the cellular structure of the above algebras is via diagrams. Diagram calculi have already been developed for many algebras: Temperley-Lieb algebras of type A (Westbury, 1995, Graham and Lehrer, 1996), Temperley-Lieb algebras of type D (Green, 1998), affine Temperley-Lieb algebras of type A (Fan and Green, 1999) and cyclotomic Temperley-Lieb algebras of type A (Rui and Xi, 2004). The idea behind this is Martin and Saleur's pictorial definition of Temperley-Lieb algebras of type A (Martin and Saleur, 1994). König and Wang (2008) gave a uniform approach to cyclotomic extensions of diagram algebras.

The purpose of this article is to introduce a cyclotomic version of Temperley-Lieb algebras of type D by means of diagrammatic generators and relations. After recalling the definition of Temperley-Lieb algebras of type D in section 2, we define the cyclotomic Temperley-Lieb algebras of type D in section 3. We prove that the cyclotomic Temperley-Lieb algebras of type D are cellular in section 4 and an explicit cellular basis is given by combinatorial methods. In section 5, we determine all the irreducible representations of these algebras and give a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebra of type D to be quasi-hereditary. In the last section, we go through a concrete example.

For simplicity, we suppose that the ground ring K is a field. It is assumed that all algebras considered in this article are finite dimensional associative K -algebras with identity, all modules are unitary, and all modules are left modules unless stated otherwise.

2 Temperley-Lieb algebras of type D

In this section we recall the definition of Temperley-Lieb algebras of type D and introduce the category of m -decorated tangles. The following figure is the Dynkin diagram of type D_n which will be used to define Temperley-Lieb algebras of type D .

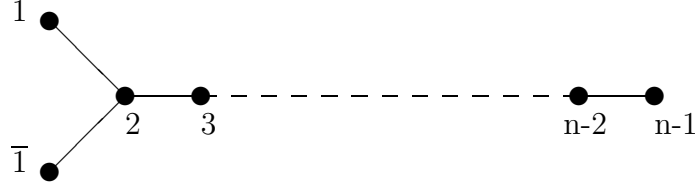


Figure 1. Dynkin diagram of type D_n

Definition 2.1 Let $n \in \mathbb{N} \geq 4$ and $\delta \in K$ be a parameter. The *Temperley-Lieb algebra of type D*, denoted by $\text{TL}_n(D)$, is an associative algebra over K with generators $E_{\bar{1}}, E_1, E_2, \dots, E_{n-1}$ subject to the following relations:

$$\begin{aligned} E_i^2 &= \delta E_i \text{ for all } i, \\ E_i E_j &= E_j E_i \text{ if } i \text{ and } j \text{ are not connected in the Dynkin diagram,} \\ E_i E_j E_i &= E_i \text{ otherwise.} \end{aligned}$$

The approach to define an algebra by diagrams has been used to understand otherwise purely abstract algebraic objects such as representations and cellular structures (in the sense of Graham and Lehrer, 1996). The category of decorated tangles was introduced in Green (1998) to study Temperley-Lieb algebras of types B and D . The idea behind this is Martin and Saleur's pictorial definition of Temperley-Lieb algebras of type A (see Martin and Saleur, 1994). We will recall the basic notions of tangles and decorated tangles in Green (1998) for later use.

Definition 2.2 Let p and q be positive integers. A *tangle* of type (p, q) is a portion of a knot diagram contained in a rectangle in the plane, consisting of arcs and closed cycles, such that the endpoints of the arcs consist of p points in the top edge of the rectangle and q points in the bottom edge.

For simplicity, a tangle of type (n, n) is said to be a tangle of type n . We refer to the boundary of a rectangle as its *frame*. Two tangles are equal if there exists an isotopy of the plane carrying one to the other such that the two diagrams can be identified when we fix the frame of each rectangle (see Freyd and Yetter, 1989 or Kauffman, 1990 for details). The endpoints of arcs are called *vertices*. The vertices in the top (respectively, bottom) edge of the frame are numbered consecutively starting with vertex number 1 at the leftmost end. Arcs in a diagram are called *horizontal arcs* if they connect two vertices sitting in the same edge of the frame; and are called *vertical arcs* if they connect two vertices sitting in the different edges.

Definition 2.3 A *decorated tangle* is a crossing-free tangle in which each arc is assigned a nonnegative integer. Any arc or closed cycle not exposed to the left face of the frame (namely, this arc or closed cycle is separated from the left face of the frame by another arc) must be assigned the integer zero.

If an arc (respectively, a closed cycle) is assigned the value s , we represent this pictorially by decorating the arc (respectively, the closed cycle) with s blobs. Each blob is marked by a hollow disc. If there are too many blobs sitting in an arc (or a closed cycle), we will mark the arc (or the closed cycle) with a representative blob, and write the value s around the blob. For any horizontal arc that links vertices i and j with $i < j$, we denote it by $\{i < j\}$. For any vertical arc that links vertices i in the top edge and j in the bottom edge, we denote it by $\{i, j\}$.

Before defining the category of decorated tangles, we first need to give the following rules for movements of blobs between connected arcs:

- (1) A blob in an arc can move freely to another arc if the two arcs share a common endpoint.
- (2) In the above movement, the number of blobs is given by adding the number of blobs from each arc.

Definition 2.4 The *category DT of decorated tangles* is defined as follows:

- (1) The objects of DT are positive integers.
- (2) The morphisms from p to q are the decorated tangles of type (p, q) .
- (3) For any $G_1 \in \text{Hom}_{\text{DT}}(p, q)$ and $G_2 \in \text{Hom}_{\text{DT}}(q, r)$, the composition $G_1 \diamond G_2$ is defined to be the concatenation of the tangle G_1 above the tangle G_2 , identifying the bottom vertices of G_1 with the top vertices of G_2 and assigning the nonnegative integers of arcs and closed cycles according to the rules for movements of blobs between connected arcs.

Remark. Note that for there to be any morphisms from p to q , it is necessary for $p + q$ to be even. A careful calculation in Green (1998) shows that $(G_1 \diamond G_2) \diamond G_3 = G_1 \diamond (G_2 \diamond G_3)$. The category-theoretic definition allows us to define an algebra of decorated tangles.

Definition 2.5 Let $n \in \mathbb{N}$. The algebra DT_n has a K -basis consisting of morphisms from n to n in DT and the multiplication is given by the composition in DT.

It is convenient to define certain named decorated tangles, $e_{\bar{1}}, e_1, e_2, \dots, e_{n-2}$ and e_{n-1} in the algebra DT_n . Define e_i , $1 \leq i \leq n-1$, to be the decorated tangle in which both the top edge and the bottom edge have a horizontal arc $\{i < i+1\}$, and the other arcs are vertical. There are no blobs on any of the arcs in e_i (for $1 \leq i \leq n-1$). Define $e_{\bar{1}}$ to be the decorated tangle obtained from e_1 by adding one blob to each of the two horizontal arcs. Now we can realize the Temperley-Lieb algebra of type D in terms of decorated tangles, and this is due to Green (1998).

Theorem 2.6 (Green, 1998). *Let $n \in \mathbb{N} \geq 4$ and $\delta \in K$ be a parameter. The algebra $\widetilde{\text{TL}}_n(D)$ has a K -basis $\{e_{\bar{1}}, e_1, e_2, \dots, e_{n-1}\}$ and the multiplication is induced from that of DT_n subject to the following relations:*

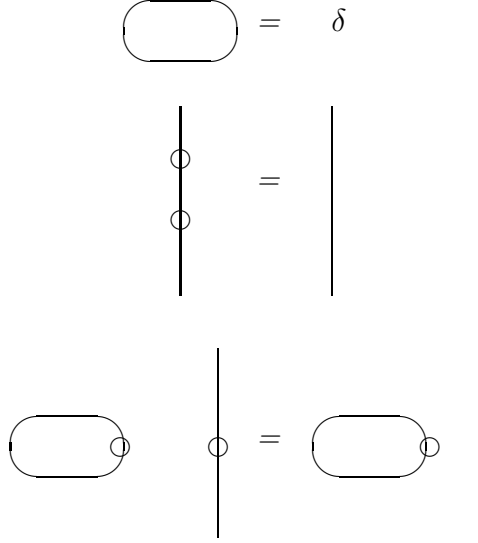


Figure 2. Relations for $\widetilde{TL}_n(D)$

There is a basis for $\widetilde{TL}_n(D)$ which is in natural bijection with elements of DT_n which have at most one blob on each arc or closed cycle, and which satisfy one of the following two mutually exclusive conditions:

(I) The diagram contains one closed cycle on which there is a blob, and there are no other closed cycles with blobs in the diagram. Also, there is at least one horizontal arc in the diagram.

(II) The diagram contains no closed cycles and the total number of blobs is even.

We say that an element of DT_n which satisfies these hypotheses is D -admissible diagram of type I or type II, depending on which of the two conditions above it satisfies.

There is an isomorphism $\rho_D : TL_n(D) \longrightarrow \widetilde{TL}_n(D)$ which takes $E_{\overline{1}}$ to $e_{\overline{1}}$ and E_i to e_i for all $1 \leq i \leq n-1$.

The first relation implies that any closed cycle with no blobs can be removed and the resulting diagram multiplied by parameter δ . The second relation gives that any arc or closed cycle with r (for $r > 1$) blobs is equivalent to the arc or closed cycle which carries $r-2$ blobs. The third relation yields that any arc or closed cycle loses its blob in the presence of a closed cycle with one blob. Using the first and third relations, all closed cycles may be removed from the resulting diagram except the last closed cycle with a blob. The following lemma gives the dimension of $TL_n(D)$.

Lemma 2.7 (Green, 1998). *Let $C(n)$ be the Catalan number $C(n) := \frac{1}{n+1} \binom{2n}{n}$. In $\widetilde{TL}_n(D)$, the number of D -admissible diagrams of type I is $C(n)-1$, and the number of D -admissible diagrams of type II is $\frac{1}{2} \binom{2n}{n}$. This is a total of $(\frac{n+3}{2})C(n)-1$ which is the dimension of $TL_n(D)$.*

To define the cyclotomic version of Temperley-Lieb algebras of type D , we need the notion of m -decorated tangles.

Definition 2.8 Let p , q , and m be positive integers. An m -decorated tangle of type (p, q) is a crossing-free tangle of type (p, q) in which each arc (and each closed cycle, if any) is assigned a pair of nonnegative integers $[r, s]$ such that r is at most $m - 1$, and such that s is assigned zero if the arc (or the closed cycle) is not exposed to the left face of the frame.

Remark. If an arc (respectively, a closed cycle) is assigned the value $[r, s]$, we represent this pictorially by decorating the arc (respectively, the closed cycle) with r dots and s blobs. Each dot is marked by a filled disc, while each blob is marked by a hollow disc. If there are too many dots or blobs sitting on an arc (or a closed cycle), we will mark the arc (or the closed cycle) with a representative dot or a representative blob, and write the value $[r, s]$ around the dot or the blob. In the case $m = 1$, a m -decorated tangle is a decorated tangle as defined before.

From now on, we make the following convention: Let $\{i < j\}$ be a horizontal arc, and assume that there are some dots or blobs sitting on the arc. There is no need to distinguish where the blobs sit, but we must distinguish the dots sitting at the left side from those at the right side of the arc. We call i the *left endpoint* and call j the *right endpoint* of the given arc $\{i < j\}$, and the dots sitting at the left (respectively, right) endpoint are called *left* (respectively, *right*) *dots* of the arc. We always assume that in an m -decorated tangle all dots are left dots. In terms of vertical arcs and closed cycles we do not define their left endpoints and right endpoints.

To define the category of m -decorated tangles, we need first to give the following rules for movements of dots and blobs between connected arcs:

- (1) A blob and a dot can interchange.
- (2) A left dot of a horizontal arc $\{i < j\}$ is equal to $m - 1$ right dots of the arc $\{i < j\}$, and conversely, a right dot of the horizontal arc is equal to $m - 1$ left dots.
- (3) A blob in an arc can move freely to another arc if the two arcs share a common endpoint. In the movement, the number of blobs is given by adding the number of blobs from each arc.
- (4) A dot in a vertical arc $\{i, j\}$ can move to another vertical arc if the two arcs share a common endpoint. In the movement, the number of dots are given directly by sum.
- (5) A right dot of a horizontal arc $\{i < j\}$ can move to another horizontal arc $\{j < k\}$ (or $\{h < j\}$), and this dot will be considered as a left dot of the arc $\{j < k\}$ (or a right dot of the arc $\{h < j\}$). Similarly, a left dot of a horizontal arc $\{i < j\}$ can move to another horizontal arc $\{k < i\}$ (or $\{i < h\}$), and this dot will be considered as a right dot of the arc $\{k < i\}$ (or a left dot of the arc $\{i < h\}$).
- (6) A dot in a vertical arc $\{i, j\}$ can move to a horizontal arc $\{k < i\}$ or $\{k < j\}$, and this dot will be regarded as a right dot on the arc $\{k < i\}$ or $\{k < j\}$. A dot in a vertical arc $\{i, j\}$ can also move to a horizontal arc $\{k > i\}$ or $\{k > j\}$, and this dot will be regarded as a left dot on the arc $\{k > i\}$ or $\{k > j\}$.
- (7) A left dot of a horizontal arc $\{i < j\}$ can move to a vertical arc $\{i, k\}$ or $\{k, i\}$. Similarly, A right dot of a horizontal arc $\{i < j\}$ can move to a vertical arc $\{j, k\}$ or $\{k, j\}$.

In the above movements, numbers of dots are reduced modulo m .

Definition 2.9 Let $m \in \mathbb{N}$. The category DT_m of m -decorated tangles is defined as follows:

- (1) The objects of DT_m are positive integers.
- (2) The morphisms from p to q are the m -decorated tangles of type (p, q) .
- (3) For any $G_1 \in \text{Hom}_{\text{DT}_m}(p, q)$ and $G_2 \in \text{Hom}_{\text{DT}_m}(q, r)$, the composition $G_1 \diamond G_2$ is defined to be the concatenation of the tangle G_1 above the tangle G_2 , identifying the bottom vertices of G_1 with the top vertices of G_2 and assigning the nonnegative integer pairs $[r, s]$ of arcs and closed cycles according to the rules for movements of dots and blobs between connected arcs.

By a careful calculation we can check that $(G_1 \diamond G_2) \diamond G_3 = G_1 \diamond (G_2 \diamond G_3)$ according to the rules for movements of dots and blobs defined as before. In the case $m = 1$, the category DT_m of 1-decorated tangles is exactly the category of decorated tangles. We end this subsection by defining an algebra of m -decorated tangles based on the previous category-theoretic definition.

Definition 2.10 Let $m, n \in \mathbb{N}$. The algebra $\text{DT}_{m,n}$ has a K -basis consisting of morphisms from n to n in DT_m , and the multiplication is given by the composition in DT_m .

3 Cyclotomic Temperley-Lieb algebras of type D

In this section we define a new class of algebras called cyclotomic Temperley-Lieb algebras of type D via diagrams. This is motivated by the work of Rui and Xi (2004), Cao and Zhu (2006) and Green (1998). We shall first focus on some special m -decorated tangles, which play a key role in describing the diagram calculus relevant to cyclotomic Temperley-Lieb algebras of type D .

Definition 3.1 Let $n \in \mathbb{N} \geq 4$ and $m \in \mathbb{N}$. An m -cyclotomic D -admissible diagram of type n is a m -decorated tangle which has at most one blob on each arc or closed cycle, and which satisfy one of the following two mutually exclusive conditions:

(I) The diagram contains one closed cycle on which there is a blob and no dots, and no other closed cycles or blobs in the diagram. Also there is at least one horizontal arc in the diagram.

(II) The diagram contains no closed cycles and the total number of blobs is even.

We say that an m -cyclotomic D -admissible diagram is of type I or type II, depending on which of the two conditions above it satisfies.

The following figure shows a typical example of 2-cyclotomic D -admissible diagrams of type $n = 4$.

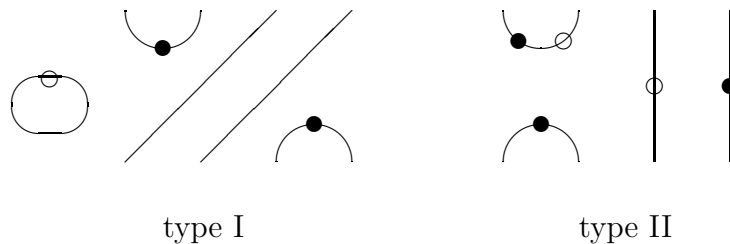


Figure 3. 2-Cyclotomic D -admissible diagrams of type $n = 4$

With the definition of m -cyclotomic D -admissible diagram, we define cyclotomic Temperley-Lieb algebras of type D as follows.

Definition 3.2 Let $n \in \mathbb{N} \geq 4$ and $m \in \mathbb{N}$. Let K be a field, and let $\delta_i \in K$ ($0 \leq i \leq m-1$) with $\delta_i \delta_0 = \delta_i$ for $1 \leq i \leq m-1$. The *cyclotomic Temperley-Lieb algebra of type D* , denoted by $\text{CTL}(D)_{m,n}$, has a K -basis consisting of m -cyclotomic D -admissible diagrams of type n and the multiplication is induced from that of $\text{DT}_{m,n}$ subject to the following relations:

$$\begin{array}{c}
 \begin{array}{c} [i,0] \\ \bullet \\ \text{---} \\ \text{---} \\ \bullet \end{array} = \delta_i \\
 \\
 \begin{array}{c} [i,2] \\ \bullet \\ \text{---} \\ \circ \end{array} = \begin{array}{c} \text{---} \\ \bullet \\ [i,0] \end{array} \\
 \\
 \begin{array}{c} [0,1] \\ \text{---} \\ \circ \end{array} \quad \begin{array}{c} [i,1] \\ \bullet \\ \text{---} \\ \circ \end{array} = \begin{array}{c} [0,1] \\ \text{---} \\ \circ \end{array} \quad \begin{array}{c} [i,0] \\ \text{---} \\ \bullet \end{array} \\
 \\
 \begin{array}{c} [i,1] \\ \bullet \\ \text{---} \\ \circ \end{array} = \delta_i \begin{array}{c} \text{---} \\ \circ \end{array} \quad (i \geq 1)
 \end{array}$$

Figure 4. Relations for $\text{CTL}(D)_{m,n}$

The first relation gives that any closed cycle with i dots and no blobs can be removed with the resulting diagram being multiplied by the parameter δ_i to compensate. The second relation implies that any arc or closed cycle with i dots and r (for $r > 1$) blobs is equivalent to the arc or closed cycle which carries i dots and $r-2$ blobs. The third relation yields that any arc or closed cycle loses its blob in the presence of a closed cycle with one blob. The fourth relation implies that any closed cycle with i (for $i \geq 1$) dots and one blob loses its dots with the resulting diagram being multiplied by the parameter δ_i to compensate. Thus, if we denote by $G_1 \circ G_2$ the diagram induced from $G_1 \diamond G_2$ (see Definition 2.9) according to the relations in Definition 3.2, we can give explicitly the expression of the multiplication:

$$G_1 \cdot G_2 = \left(\prod_{i=1}^{m-1} \delta_i^{n(\bar{i}, G_1, G_2)^+} \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, G_1, G_2)^-} \right) G_1 \circ G_2,$$

where $n(\bar{i}, G_1, G_2)^+$ is the number of closed cycles with i dots and one blob in $G_1 \diamond G_2$ (see Definition 2.9), and $n(\bar{i}, G_1, G_2)^-$ is the number of closed cycles with i dots and no blobs in $G_1 \diamond G_2$. We can check that $(G_1 \cdot G_2) \cdot G_3 = G_1 \cdot (G_2 \cdot G_3)$. In fact

$$(G_1 \cdot G_2) \cdot G_3 = (\delta_0^{N_0} \prod_{i=1}^{m-1} \delta_i^{N_i})(G_1 \circ G_2) \circ G_3,$$

$$G_1 \cdot (G_2 \cdot G_3) = (\delta_0^{N'_0} \prod_{i=1}^{m-1} \delta_i^{N'_i})G_1 \circ (G_2 \circ G_3),$$

where $N_0 = n(\bar{0}, G_1, G_2)^- + n(\bar{0}, G_1 \circ G_2, G_3)^-$, $N_i = n(\bar{i}, G_1, G_2)^+ + n(\bar{i}, G_1, G_2)^- + n(\bar{i}, G_1 \circ G_2, G_3)^+ + n(\bar{i}, G_1 \circ G_2, G_3)^-$ and $N'_0 = n(\bar{0}, G_2, G_3)^- + n(\bar{0}, G_1, G_2 \circ G_3)^-$, $N'_i = n(\bar{i}, G_2, G_3)^+ + n(\bar{i}, G_2, G_3)^- + n(\bar{i}, G_1, G_2 \circ G_3)^+ + n(\bar{i}, G_1, G_2 \circ G_3)^-$. Since $(G_1 \circ G_2) \circ G_3$ is induced from $(G_1 \diamond G_2) \diamond G_3$ and $G_1 \circ (G_2 \circ G_3)$ is induced from $G_1 \diamond (G_2 \diamond G_3)$ according to the relations in Definition 3.2, we know that $(G_1 \circ G_2) \circ G_3 = G_1 \circ (G_2 \circ G_3)$ from $(G_1 \diamond G_2) \diamond G_3 = G_1 \diamond (G_2 \diamond G_3)$. Note that the total number of closed cycles with i (for $i \geq 1$) dots and one blob and closed cycles with i (for $i \geq 1$) dots and no blobs in $G_1 \diamond G_2$ and $(G_1 \circ G_2) \diamond G_3$ is equal to the total number of closed cycles with i (for $i \geq 1$) dots and one blob and closed cycles with i (for $i \geq 1$) dots and no blobs in $G_2 \diamond G_3$ and $G_1 \diamond (G_2 \circ G_3)$. Thus $N_i = N'_i$ (for $1 \leq i \leq m-1$). If there exists $N_i \neq 0$ for some $1 \leq i \leq m-1$, then $\delta_0^{N_0} \prod_{i=1}^{m-1} \delta_i^{N_i} = \delta_0^{N'_0} \prod_{i=1}^{m-1} \delta_i^{N'_i}$ from $\delta_i \delta_0 = \delta_i$ for $1 \leq i \leq m-1$. If not, because the total number of closed cycles with no dots and one blob and closed cycles with no dots and no blobs in $G_1 \diamond G_2$ and $(G_1 \circ G_2) \diamond G_3$ is equal to the total number of closed cycles with no dots and one blob and closed cycles with no dots and no blobs in $G_2 \diamond G_3$ and $G_1 \diamond (G_2 \circ G_3)$, we have $\delta_0^{N_0} = \delta_0^{N'_0}$.

Cyclotomic Temperley-Lieb algebras of type D are a generalization of the usual Temperley-Lieb algebras of type D (see Graham, 1995, Green, 1998) on the one hand and have cyclotomic Temperley-Lieb algebras of type A (see Rui and Xi, 2004) with parameter $\delta_0 = 1$ as a class of subalgebras on the other hand. If $m = 1$, then $\text{CTL}(D)_{m,n}$ is the usual Temperley-Lieb algebra of type D with K -dimension $(\frac{n+3}{2})C(n) - 1$. From this, it is clear that the cyclotomic Temperley-Lieb algebra of type D has K -dimension $m^n((\frac{n+3}{2})C(n) - 1)$. Note that cyclotomic Temperley-Lieb algebras of type D are very different from cyclotomic blob algebras $\text{CB}_{m,n}$ (see Cao and Zhu, 2006 for the definition) not only in their basis diagrams but also in their parameters and relations for multiplication. The diagrams in $\text{CTL}(D)_{m,n}$ may have a closed circle with one blob while the diagrams in $\text{CB}_{m,n}$ must not contain any closed circles. And the total number of blobs in m -cyclotomic D -admissible diagrams of type II must be even, while the diagrams in $\text{CB}_{m,n}$ do not have this restriction. In addition, $\text{CTL}(D)_{m,n}$ have m parameters while $\text{CB}_{m,n}$ have $2m$ parameters.

It is convenient for us to introduce the following notations for later use. Denote by $Q(n, k)$ the set of all m -cyclotomic D -admissible diagrams in which both the top edge and the bottom edge have n vertices and k horizontal arcs, by $Q^+(n, k)$ ($1 \leq k \leq \lfloor \frac{n}{2} \rfloor$) the subset of $Q(n, k)$ that consists of m -cyclotomic D -admissible diagrams of type I

and by $Q^-(n, k)$ ($0 \leq k \leq \lfloor \frac{n}{2} \rfloor$) the subset of $Q(n, k)$ that consists of m -cyclotomic D -admissible diagrams of type II. In the case $k = \frac{n}{2}$, we define $Q_1^-(n, \frac{n}{2})$ (respectively, $Q_2^-(n, \frac{n}{2})$) to be the subset of $Q^-(n, \frac{n}{2})$ that consists of diagrams which have even (respectively, odd) blobs both in the upper part and in the lower part of the diagrams. We define $P^+(n, k)$ (respectively, $P^-(n, k)$) to be the vector space spanned by all m -cyclotomic D -admissible diagrams in $Q^+(n, k)$ (respectively, $Q^-(n, k)$). In the case $k = \frac{n}{2}$, we define $P_1^-(n, \frac{n}{2})$ (respectively, $P_2^-(n, \frac{n}{2})$) to be the vector space spanned by all m -cyclotomic D -admissible diagrams in $Q_1^-(n, \frac{n}{2})$ (respectively, $Q_2^-(n, \frac{n}{2})$). We also define certain named m -cyclotomic D -admissible diagrams $\mathcal{T}_1, \dots, \mathcal{T}_n$ in the algebra $\text{CTL}(D)_{m,n}$.

Define \mathcal{T}_i (for $1 \leq i \leq n$) to be the m -cyclotomic D -admissible diagram in which all arcs are vertical with no blobs, the i -th vertical arc carries one dot, and there are no other dots.

In the case $n = 4$, the m -cyclotomic D -admissible diagrams $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and \mathcal{T}_4 are as shown in the following figure.

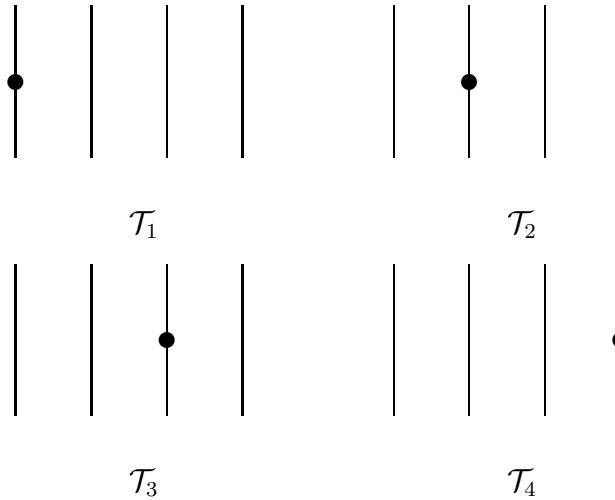


Figure 5. m -Cyclotomic D -admissible diagrams $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and \mathcal{T}_4

4 Cellular structure of $\text{CTL}(D)_{m,n}$

In this section, we investigate the cellular structure of cyclotomic Temperley-Lieb algebras of type D . After recalling the definition of cellular algebras, we construct a cellular basis for $\text{CTL}(D)_{m,n}$ using combinatorial methods.

Definition 4.1 (Graham and Lehrer, 1996). An associative K -algebra A is called a *cellular algebra* with cell datum (Λ, M, C, i) if the following conditions are satisfied:

(C1) The finite set Λ is partially ordered. Associated with each $\lambda \in \Lambda$ there is a finite indexing set $M(\lambda)$. The algebra A has a K -basis $C_{S,T}^\lambda$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.

(C2) The map i is a K -linear anti-automorphism of A with $i^2 = id$ which sends $C_{S,T}^\lambda$ to $C_{T,S}^\lambda$ for all $\lambda \in \Lambda$ and all S and T in $M(\lambda)$.

(C3) For each $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and each $a \in A$, the product $aC_{S,T}^\lambda$ can be written as $(\sum_{U \in M(\lambda)} r_a(U, S)C_{U,T}^\lambda) + r'$, where r' is a linear combination of basis elements with upper index μ strictly smaller than λ , and where the coefficients $r_a(U, S) \in K$ are independent of T .

The basis $\{C_{S,T}^\lambda \mid \lambda \in \Lambda \text{ and } S, T \in M(\lambda)\}$ satisfying the above condition is called a *cellular basis* of A . The K -linear anti-automorphism i of A with $i^2 = id$ is called an *involution*. Whether an algebra is cellular or not depends on the choice of the involution i . A cellular algebra can have more than one cellular basis and both the poset Λ and the indexing sets $M(\lambda)$ may vary dramatically between different cellular bases of the same algebra. The size of the poset Λ can also be different for different cellular bases of an algebra (see König and Xi, 1999b). We now recall the ring theoretic definition of cellular algebras, which is equivalent to the original one.

Definition 4.2 (König and Xi, 1998). Let A be a K -algebra with an involution i . A two-sided ideal J of A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $W \subset J$ such that there is an isomorphism of A -bimodules $\alpha : J \simeq W \otimes_K i(W)$ making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & W \otimes_K i(W) \\ i \downarrow & & \downarrow x \otimes y \mapsto i(y) \otimes i(x) \\ J & \xrightarrow{\alpha} & W \otimes_K i(W) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=1}^j J'_l$ gives a chain of two-sided ideals of A : $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \dots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1} . We call this chain a *cell chain* for the cellular algebra A .

Remark. It has been shown that the class of cellular algebras includes a number of well-known algebras such as Ariki-Koike algebras (see Ariki and Koike, 1994), Brauer algebras (see Graham and Lehrer, 1996, König and Xi, 1999a), Jones' annular algebras (see Graham and Lehrer, 1996), and partition algebras (see Xi, 1999) as well as Birman-Wenzl algebras (see Xi, 2000).

We recall a simple example of cellular algebras in Rui and Xi (2004) for later use. Take $G_{m,n}$ to be the K -subalgebra of the cyclotomic Temperley-Lieb algebra of type D generated by $\mathcal{T}_1, \dots, \mathcal{T}_n$. $G_{m,n}$ is isomorphic to the group algebra of the abelian group $\bigoplus_{i=1}^n \mathbb{Z}/m\mathbb{Z}$. Assume that K is a splitting field of $x^m - 1$, then we can write $\mathcal{T}_i^m = 1$ as $\prod_{j=1}^m (\mathcal{T}_i - \xi_j) = 0$ for some $\xi_1, \dots, \xi_m \in K$. Let $\Lambda(m, n) = \{(i_1, \dots, i_n) \mid 1 \leq i_k \leq m\}$ for $n \geq 1$, and assume that in the case $n = 0$ the set $\Lambda(m, n)$ consists of only one element \emptyset . We define $(i_1, \dots, i_n) \preceq (j_1, \dots, j_n)$ if and only if $i_k \leq j_k$ for all $1 \leq k \leq n$.

For each $I = (i_1, \dots, i_n)$, define $C_{1,1}^I = \prod_{j=1}^n \prod_{l=i_j+1}^m (\mathcal{T}_j - \xi_l)$, where the product over empty set is assumed to be 1. We observe that $\{C_{1,1}^I \mid I \in \Lambda(m, n)\}$ is a cellular basis of the algebra $G_{m,n}$ with respect to the identity involution. In order to construct a cellular basis for $\text{CTL}(D)_{m,n}$ we introduce two further notions.

Definition 4.3 An m -decorated dangle of type (n, k) is a crossing-free diagram consisting of n vertices $\{1, \dots, n\}$, k horizontal arcs, $n - 2k$ vertical lines and some closed cycles which satisfy the following conditions.

- (1) Each horizontal arc (and each closed cycle, if any) carries at most $m - 1$ dots, and each vertical line does not carry any dots.
- (2) Only the leftmost vertical line and the horizontal arcs (and the closed cycles, if any) which appear to the left of the leftmost vertical line and to the outermost of any nested horizontal arcs may carry at most one blob.

Definition 4.4 An m -cyclotomic D -admissible dangle of type (n, k) is an m -decorated dangle of type (n, k) which satisfies one of the following two mutually exclusive conditions:

- (I) The diagram contains one closed cycle on which there is a blob and there are no dots, and no other closed cycles or blobs in the diagram. Also there is at least one horizontal arc in the diagram.
- (II) The diagram contains no closed cycles and the total number of blobs is even if $k \neq \frac{n}{2}$.

We say that an m -cyclotomic D -admissible dangle is of type I or type II, depending on which of the two conditions above it satisfies.

We denote by $D(n, k)$ the set of all m -cyclotomic D -admissible dangles of type (n, k) , by $D^+(n, k)$ ($1 \leq k \leq [\frac{n}{2}]$) the subset of $D(n, k)$ that consists of m -cyclotomic D -admissible dangles of type I and by $D^-(n, k)$ ($0 \leq k \leq [\frac{n}{2}]$) the subset of $D(n, k)$ that consists of m -cyclotomic D -admissible dangles of type II. In the case $k = \frac{n}{2}$, we define $D_1^-(n, \frac{n}{2})$ (respectively, $D_2^-(n, \frac{n}{2})$) to be the subset of $D^-(n, \frac{n}{2})$ that consists of dangles with an even (respectively, odd) number of blobs in total. We define $V^+(n, k)$ (respectively, $V^-(n, k)$) to be the vector space spanned by all m -cyclotomic D -admissible dangles in $D^+(n, k)$ (respectively, $D^-(n, k)$). In the case $k = \frac{n}{2}$, we define $V_1^-(n, \frac{n}{2})$ (respectively, $V_2^-(n, \frac{n}{2})$) to be the vector space spanned by all m -cyclotomic D -admissible dangles in $D_1^-(n, \frac{n}{2})$ (respectively, $D_2^-(n, \frac{n}{2})$). The following lemma shows a close relationship between $P^+(n, k)$ (respectively, $P^-(n, k)$) and $V^+(n, k)$ (respectively, $V^-(n, k)$).

Lemma 4.5 *There are three K -module isomorphisms:*

- (1) $P^+(n, k) \simeq V^+(n, k) \otimes_K V^+(n, k) \otimes_K G_{m, n-2k}$ for $1 \leq k \leq [\frac{n}{2}]$.
- (2) $P^-(n, k) \simeq V^-(n, k) \otimes_K V^-(n, k) \otimes_K G_{m, n-2k}$ for $1 \leq k \leq [\frac{n}{2}]$ and $k \neq \frac{n}{2}$.
- (3) $P_i^-(n, k) \simeq V_i^-(n, k) \otimes_K V_i^-(n, k) \otimes_K G_{m, 0}$ ($i = 1, 2$).

Proof. We first prove (1). Suppose that $G \in P^+(n, k)$. Then G has $n - 2k$ vertical arcs. Let m_i be the number of dots in the i -th vertical arc and let $\omega = \mathcal{T}_1^{m_1} \dots \mathcal{T}_{n-2k}^{m_{n-2k}}$.

Clearly, we have $\omega \in G_{m,n-2k}$. Cutting all vertical arcs and omitting all dots in the vertical arcs, we can divide the diagram G into two half diagrams, each with a closed cycle carrying one blob. Denote by G_1 the upper part, and by G_2 the lower part. Consequently, we can write G uniquely as $G_1 \otimes G_2 \otimes \omega$, which belongs to $V^+(n, k) \otimes_K V^+(n, k) \otimes_K G_{m,n-2k}$. Conversely, given such an expression $G_1 \otimes G_2 \otimes \omega$, we have a unique diagram G with G_1 on the top and G_2 on the bottom, the ends being joined in the unique way which creates a crossing-free diagram. Omitting one closed cycle in G , we get a m -cyclotomic D -admissible diagram of type I in $P^+(n, k)$. This proves (1).

The proof of (2) is similar to that of (1). The difference is that before cutting the leftmost vertical arc we first omit the blob (if any) on it. If the total number of blobs in the horizontal arcs of G_1 (respectively, G_2) is odd, then the leftmost vertical line will be assigned one blob such that G_1 (respectively, G_2) belongs to $V^-(n, k)$.

In case of $k = \frac{n}{2}$, there are no vertical arcs in the m -cyclotomic D -admissible diagrams. Thus the proof of (3) is straightforward. \square

As a result, we have the following equivalent basis of $\text{CTL}(D)_{m,n}$. However, this basis is usually not a cellular basis. Denote by $\overline{G}_{m,n-2k}$ the set of a K -basis of $G_{m,n-2k}$. Let $V = \{v_1 \otimes v_2 \otimes \omega \mid v_1, v_2 \in D^+(n, k), \omega \in \overline{G}_{m,n-2k}, 1 \leq k \leq [\frac{n}{2}]\} \cup \{v_1 \otimes v_2 \otimes \omega \mid v_1, v_2 \in D^-(n, k), \omega \in \overline{G}_{m,n-2k}, 0 \leq k \leq [\frac{n}{2}] \text{ and } k \neq \frac{n}{2}\}$ and $V' = \{v_1 \otimes v_2 \otimes 1 \mid v_1, v_2 \in D_1^-(n, \frac{n}{2})\} \cup \{v_1 \otimes v_2 \otimes 1 \mid v_1, v_2 \in D_2^-(n, \frac{n}{2})\}$.

Corollary 4.6 *The set V (respectively, $V \cup V'$) constitutes a basis for $\text{CTL}(D)_{m,n}$ if n is odd (respectively, even).*

Now we describe the cellular structure of $\text{CTL}(D)_{m,n}$. Recall that $\Lambda(m, n-2k) = \{(i_1, \dots, i_{n-2k}) \mid 1 \leq i_j \leq m\}$. If n is odd, let $\Lambda_{m,n} = \{(\tilde{k}, I)^+, (k, J)^- \mid 1 \leq \tilde{k} \leq [\frac{n}{2}], 0 \leq k \leq [\frac{n}{2}], I \in \Lambda(m, n-2\tilde{k}), J \in \Lambda(m, n-2k)\}$. We define a partial order " \leq " on $\Lambda_{m,n}$:

- (1) $(\tilde{k}, I)^+ \leq (k, J)^-$ for any \tilde{k}, k, I, J ;
- (2) $(k_1, I_1)^+ \leq (k_2, I_2)^+$ if $k_1 > k_2$;
- (3) $(\tilde{k}, I_1)^+ \leq (\tilde{k}, I_2)^+$ if $I_1 \preceq I_2$;
- (4) $(k_1, J_1)^- \leq (k_2, J_2)^-$ if $k_1 > k_2$;
- (5) $(k, J_1)^- \leq (k, J_2)^-$ if $J_1 \preceq J_2$;

From the relations (2) and (3) we know that $\{(\tilde{k}, I)^+ \mid 1 \leq \tilde{k} \leq [\frac{n}{2}], I \in \Lambda(m, n-2\tilde{k})\}$ is a partially ordered set, as is $\{(k, J)^- \mid 0 \leq k \leq [\frac{n}{2}], J \in \Lambda(m, n-2k)\}$ from relations (4) and (5). Note that $\Lambda_{m,n}$ is in fact a disjoint union of these two subsets with the former strictly smaller than the latter by relation (1). From the partial orderings of $\{(\tilde{k}, I)^+ \mid 1 \leq \tilde{k} \leq [\frac{n}{2}], I \in \Lambda(m, n-2\tilde{k})\}$ and $\{(k, J)^- \mid 0 \leq k \leq [\frac{n}{2}], J \in \Lambda(m, n-2k)\}$ we can check that \leq is reflexive, transitive and antisymmetric, thus $(\Lambda_{m,n}, \leq)$ is a finite partially ordered set. If n is even, let $\Lambda_{m,n} = \{(\tilde{k}, I)^+, (\frac{n}{2}, \emptyset)_1^-, (\frac{n}{2}, \emptyset)_2^-, (k, J)^- \mid 1 \leq \tilde{k} \leq [\frac{n}{2}], 0 \leq k \leq [\frac{n}{2}] - 1, I \in \Lambda(m, n-2\tilde{k}), J \in \Lambda(m, n-2k)\}$. In this case, we substitute $(\tilde{k}, I)^+ \leq (\frac{n}{2}, \emptyset)_i^-$ ($i = 1, 2$) $\leq (k, J)^-$ for relation (1) above and retain

the four other relations to define a partial order on $\Lambda_{m,n}$. Similarly, we can prove that $(\Lambda_{m,n}, \leq)$ is a finite partially ordered set.

For each $(\tilde{k}, I)^+ \in \Lambda_{m,n}$, define $M((\tilde{k}, I)^+) = \{(v, I) \mid v \in D^+(n, \tilde{k})\}$. Similarly, define $M((k, J)^-) = \{(v, J) \mid v \in D^-(n, k)\}$ for each $(k, J)^- \in \Lambda_{m,n}$. In the case $k = \frac{n}{2}$, define $M((\frac{n}{2}, \emptyset)_i^-) = \{(v, \emptyset) \mid v \in D_i^-(n, \frac{n}{2})\}$ ($i = 1, 2$).

In the following we use the cellular bases $\{C_{1,1}^I \mid I \in \Lambda(m, n-2\tilde{k})\}$ of $G_{m,n-2\tilde{k}}$ and $\{C_{1,1}^J \mid J \in \Lambda(m, n-2k)\}$ of $G_{m,n-2k}$ to construct a cellular basis for $\text{CTL}(D)_{m,n}$. For each $(\tilde{k}, I)^+ \in \Lambda_{m,n}$ and $v_1, v_2 \in D^+(n, \tilde{k})$, let $C_{(v_1, I), (v_2, I)}^{(\tilde{k}, I)^+} = v_1 \otimes v_2 \otimes C_{1,1}^I$. Similarly, let $C_{(v_1, J), (v_2, J)}^{(k, J)^-} = v_1 \otimes v_2 \otimes C_{1,1}^J$ for each $(k, J)^- \in \Lambda_{m,n}$ and $v_1, v_2 \in D^-(n, k)$. In the case $k = \frac{n}{2}$, let $C_{(v_1, \emptyset), (v_2, \emptyset)}^{(\frac{n}{2}, \emptyset)_i^-} = v_1 \otimes v_2 \otimes C_{1,1}^\emptyset$ for each $v_1, v_2 \in D_i^-(n, \frac{n}{2})$ ($i = 1, 2$). Let

$$\begin{aligned} C(\tilde{k}, I)^+ &= \left\{ C_{(v_1, I), (v_2, I)}^{(\tilde{k}, I)^+} \mid v_1, v_2 \in D^+(n, \tilde{k}) \right\} \\ C(k, J)^- &= \left\{ C_{(v_1, J), (v_2, J)}^{(k, J)^-} \mid v_1, v_2 \in D^-(n, k) \right\} \\ C\left(\frac{n}{2}, \emptyset\right)_i^- &= \left\{ C_{(v_1, \emptyset), (v_2, \emptyset)}^{(\frac{n}{2}, \emptyset)_i^-} \mid v_1, v_2 \in D_i^-(n, \frac{n}{2}) \right\} \quad (i = 1, 2) \end{aligned}$$

By Corollary 4.6, the set

$$\left(\bigcup_{\tilde{k}=1}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{I \in \Lambda(m, n-2\tilde{k})} C(\tilde{k}, I)^+ \right) \cup \left(\bigcup_{k=0}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{J \in \Lambda(m, n-2k)} C(k, J)^- \right)$$

forms a basis for $\text{CTL}(D)_{m,n}$ if n is odd and the set

$$\left(\bigcup_{\tilde{k}=1}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{I \in \Lambda(m, n-2\tilde{k})} C(\tilde{k}, I)^+ \right) \cup \left(\bigcup_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \bigcup_{J \in \Lambda(m, n-2k)} C(k, J)^- \right) \cup \left(\bigcup_{i=1}^2 C\left(\frac{n}{2}, \emptyset\right)_i^- \right)$$

forms a basis for $\text{CTL}(D)_{m,n}$ if n is even.

The involution i corresponds to top-bottom inversion of an m -cyclotomic D -admissible diagram. The following lemma shows in detail how i acts on $\text{CTL}(D)_{m,n}$.

Lemma 4.7 *Let i be the K -linear anti-automorphism of $\text{CTL}(D)_{m,n}$ which corresponds to top-bottom inversion of an m -cyclotomic D -admissible diagram. Then we have the following.*

- (1) i sends $v_1 \otimes v_2 \otimes C_{1,1}^I$ to $v_2 \otimes v_1 \otimes C_{1,1}^I$ for all $v_1, v_2 \in D^+(n, \tilde{k})$ and $C_{1,1}^I \in \overline{G}_{m, n-2\tilde{k}}$.
- (2) i sends $v_1 \otimes v_2 \otimes C_{1,1}^J$ to $v_2 \otimes v_1 \otimes C_{1,1}^J$ for all $v_1, v_2 \in D^-(n, k)$ and $C_{1,1}^J \in \overline{G}_{m, n-2k}$.
- (3) i sends $v_1 \otimes v_2 \otimes C_{1,1}^\emptyset$ to $v_2 \otimes v_1 \otimes C_{1,1}^\emptyset$ for all $v_1, v_2 \in D_i^-(n, \frac{n}{2})$ ($i = 1, 2$) if $k = \frac{n}{2}$.

Proof. We first prove (1). Suppose $G = v_1 \otimes v_2 \otimes C_{1,1}^I \in C(\tilde{k}, I)^+$ which has a closed cycle with one blob, \tilde{k} horizontal arcs $\{i_{11} < i_{12}\}, \{i_{21} < i_{22}\}, \dots, \{i_{\tilde{k}1} < i_{\tilde{k}2}\}$ in the top edge, \tilde{k} horizontal arcs $\{j_{11} < j_{12}\}, \{j_{21} < j_{22}\}, \dots, \{j_{\tilde{k}1} < j_{\tilde{k}2}\}$ in the bottom edge and $n - 2\tilde{k}$ vertical arcs $\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_{\tilde{k}}, j_{\tilde{k}}\}$. Then v_1 is the m -cyclotomic D -admissible dangle of type I with \tilde{k} horizontal arcs $\{i_{11} < i_{12}\}, \{i_{21} < i_{22}\}, \dots, \{i_{\tilde{k}1} < i_{\tilde{k}2}\}$ and $n - 2\tilde{k}$ vertical lines $i_1, i_2, \dots, i_{\tilde{k}}$, and v_2 is the m -cyclotomic D -admissible dangle of type I with \tilde{k} horizontal arcs $\{j_{11} < j_{12}\}, \{j_{21} < j_{22}\}, \dots, \{j_{\tilde{k}1} < j_{\tilde{k}2}\}$ and $n - 2\tilde{k}$ vertical lines $j_1, j_2, \dots, j_{\tilde{k}}$. Let G' be the top-bottom inversion of G . We can describe G' explicitly, that is, G' has a closed cycle with one blob, \tilde{k} horizontal arcs $\{j_{11} < j_{12}\}, \{j_{21} < j_{22}\}, \dots, \{j_{\tilde{k}1} < j_{\tilde{k}2}\}$ in the top edge, \tilde{k} horizontal arcs $\{i_{11} < i_{12}\}, \{i_{21} < i_{22}\}, \dots, \{i_{\tilde{k}1} < i_{\tilde{k}2}\}$ in the bottom edge and $n - 2\tilde{k}$ vertical arcs $\{j_1, i_1\}, \{j_2, i_2\}, \dots, \{j_{\tilde{k}}, i_{\tilde{k}}\}$. Using the cutting method in the proof of Lemma 4.5 we know that $G' = v_2 \otimes v_1 \otimes C_{1,1}^I$.

The proof of (2) is more complicated. Suppose $G = v_1 \otimes v_2 \otimes C_{1,1}^J \in C(k, J)^-$, where v_1 is the m -cyclotomic D -admissible dangle of type II with k horizontal arcs $\{i_{11} < i_{12}\}, \{i_{21} < i_{22}\}, \dots, \{i_{k1} < i_{k2}\}$ and $n - 2k$ vertical lines i_1, i_2, \dots, i_k and v_2 is the m -cyclotomic D -admissible dangle of type II with k horizontal arcs $\{j_{11} < j_{12}\}, \{j_{21} < j_{22}\}, \dots, \{j_{k1} < j_{k2}\}$ and $n - 2k$ vertical lines j_1, j_2, \dots, j_k . Suppose the total number of blobs in v_1 is $2s$ and the total number of blobs in v_2 is $2t$, where $s, t \in \mathbb{N} \cup \{0\}$. Then there are two cases for v_1 : One is that there is a blob on the vertical line i_1 and the total number of blobs on the horizontal arcs is $2s - 1$ ($s \geq 1$), and the other is that there are no blobs on the vertical line i_1 and the total number of blobs on the horizontal arcs is $2s$. Similarly there are two cases for v_2 . So there are four cases of G for us to consider. In the case v_1 has a blob on the vertical line i_1 and $2s - 1$ blobs on its horizontal arcs and v_2 has a blob on the vertical line j_1 and $2t - 1$ blobs on its horizontal arcs, there are no blobs on the leftmost vertical arc of G and the total number of blobs on the horizontal arcs is $2(s + t - 1)$. Let G' be the top-bottom inversion of G . Then G' has no blobs on the leftmost vertical arc $\{j_1, i_1\}$, $2s - 1$ blobs on the horizontal arcs $\{j_{11} < j_{12}\}, \{j_{21} < j_{22}\}, \dots, \{j_{k1} < j_{k2}\}$ in the top edge and $2t - 1$ blobs on the horizontal arcs $\{i_{11} < i_{12}\}, \{i_{21} < i_{22}\}, \dots, \{i_{k1} < i_{k2}\}$ in the bottom edge. Using the cutting method in the proof of Lemma 4.5 we know that $G' = v_2 \otimes v_1 \otimes C_{1,1}^J$. The other three cases can be shown similarly. So i sends $v_1 \otimes v_2 \otimes C_{1,1}^J$ to $v_2 \otimes v_1 \otimes C_{1,1}^J$ for all $v_1, v_2 \in D^-(n, k)$ and $C_{1,1}^J \in \overline{G}_{m, n-2k}$.

In case of $k = \frac{n}{2}$, the proof is straightforward since there is no vertical arc in the m -cyclotomic D -admissible diagrams. \square

The following theorem shows that the datum $(\Lambda_{m,n}, M, C, i)$ makes the algebra $\text{CTL}(D)_{m,n}$ into a cellular algebra.

Theorem 4.8 *Let K be a splitting field of $x^m - 1$. Then the cyclotomic Temperley-Lieb algebra of type D over K is a cellular algebra with cell datum $(\Lambda_{m,n}, M, C, i)$.*

Proof. By the above construction of the datum $(\Lambda_{m,n}, M, C, i)$, it is clear that the first two conditions of Definition 4.1 are satisfied. Now, we will verify the condition (C3)

of the definition. We consider the product $a \cdot C_{S,T}^\lambda$ for each $\lambda \in \Lambda_{m,n}$ and $S, T \in M(\lambda)$ and each $a \in \text{CTL}(D)_{m,n}$. Note that all m -cyclotomic D -admissible diagrams of type n form a free K -basis of $\text{CTL}(D)_{m,n}$, so we only have to consider the case when a is a basis element. We denote by $\text{CTL}(D)_{m,n}^{\leq \lambda}$ (respectively, $\text{CTL}(D)_{m,n}^{< \lambda}$, $\text{CTL}(D)_{m,n}^\lambda$) the K -subspace of $\text{CTL}(D)_{m,n}$ spanning by all basis elements with upper index smaller than (respectively, strictly smaller than, equal to) λ .

When n is odd, we have the following four cases to prove.

(1) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q^+(n, k)$ and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^I \in C(\tilde{k}, I)^+$, let $y = B \cdot G$. We need to consider the horizontal arcs in B and G . It is immediate that $y \in \text{CTL}(D)_{m,n}^{<(\tilde{k}, I)^+}$ when $k > \tilde{k}$ since there are at least k horizontal arcs in each edge in y . The case $k \leq \tilde{k}$ is more subtle. It is clear that $y \in \text{CTL}(D)_{m,n}^{\leq(\tilde{k}, I)^+}$. Suppose

$$y \equiv \left(\prod_{i=1}^{m-1} \delta_i^{n(\tilde{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\tilde{i}, B, G)^-} \right) B' \otimes G_2 \otimes \omega' C_{1,1}^I$$

(mod $\text{CTL}(D)_{m,n}^{<(\tilde{k}, I)^+}$) $\in \text{CTL}(D)_{m,n}^{\leq(\tilde{k}, I)^+}$, where $B' \in D^+(n, \tilde{k})$, $\omega' \in G_{m, n-2\tilde{k}}$. We observe that the eliminated closed cycles are completely determined by the horizontal arcs in B_2 and G_1 . Hence the coefficients $\delta_i^{n(\tilde{i}, B, G)^+}$ and $\delta_i^{n(\tilde{i}, B, G)^-}$ depend only on B_2 and G_1 and the closed cycles in B and G . Note that B' is determined only by B_1 , B_2 and G_1 , and ω' depends on B_2 , ω , G_1 and $C_{1,1}^I$, thus B' and ω' are independent of G_2 . Write $\omega' = \prod_{j=1}^{n-2\tilde{k}} \mathcal{T}_j^{k_j}$ for some $0 \leq k_j \leq m-1$, $1 \leq j \leq n-2\tilde{k}$. By a careful calculation, we know that $\omega' C_{1,1}^I \equiv \prod_{j=1}^{n-2\tilde{k}} \xi_{i_j}^{k_j} C_{1,1}^I$ (mod $G_{m, n-2k}^{<I}$), where $G_{m, n-2k}^{<I}$ is the K -subspace of $G_{m, n-2\tilde{k}}$ spanned by $C_{1,1}^J$ with J strictly smaller than I . Note also that the coefficient $\prod_{j=1}^{n-2\tilde{k}} \xi_{i_j}^{k_j}$ is independent of G_2 . Therefore,

$$y \equiv \left(\prod_{i=1}^{m-1} \delta_i^{n(\tilde{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\tilde{i}, B, G)^-} \prod_{j=1}^{n-2\tilde{k}} \xi_{i_j}^{k_j} \right) B' \otimes G_2 \otimes C_{1,1}^I$$

(mod $\text{CTL}(D)_{m,n}^{<(\tilde{k}, I)^+}$). By the above argument, both B' and the coefficient of $B' \otimes G_2 \otimes C_{1,1}^I$ are independent of G_2 .

(2) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q^-(n, k)$ and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^I \in C(\tilde{k}, I)^+$, let $y = B \cdot G$. It is clear that $y \in \text{CTL}(D)_{m,n}^{<(\tilde{k}, I)^+}$ when $k > \tilde{k}$. In the case $k \leq \tilde{k}$, we have $y \in \text{CTL}(D)_{m,n}^{\leq(\tilde{k}, I)^+}$. An argument similar to (1) shows that

$$y \equiv \left(\prod_{i=1}^{m-1} \delta_i^{n(\tilde{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\tilde{i}, B, G)^-} \prod_{j=1}^{n-2\tilde{k}} \xi_{i_j}^{k_j} \right) B' \otimes G_2 \otimes C_{1,1}^I$$

(mod $\text{CTL}(D)_{m,n}^{<(\tilde{k}, I)^+}$), where both B' and the coefficient of $B' \otimes G_2 \otimes C_{1,1}^I$ are independent of G_2 .

(3) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q^+(n, k_1)$ and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^J \in C(k_2, J)^-$, it is always true that $y = B \cdot G \in \text{CTL}(D)_{m,n}^{<(k_2, J)^-}$ since there is always a closed cycle in y .

(4) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q^-(n, k_1)$ and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^J \in C(k_2, J)^-$, let $y = B \cdot G$. It is clear that $y \in \text{CTL}(D)_{m,n}^{<(k_2, J)^-}$ when $k_1 > k_2$. In the case $k_1 \leq k_2$, we also have $y \in \text{CTL}(D)_{m,n}^{<(k_2, J)^-}$ if the horizontal arcs in B_2 and G_1 form an interior closed cycle with one blob. Otherwise we have $y \in \text{CTL}(D)_{m,n}^{\leq(k_2, J)^-}$. An argument similar to (1) shows that

$$y \equiv \left(\prod_{i=1}^{m-1} \delta_i^{n(\bar{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, B, G)^-} \prod_{j=1}^{n-2k_2} \xi_{i_j}^{k_j} \right) B' \otimes G_2 \otimes C_{1,1}^J$$

(mod $\text{CTL}(D)_{m,n}^{<(k_2, J)^-}$), where both B' and the coefficient of $B' \otimes G_2 \otimes C_{1,1}^J$ are independent of G_2 .

When n is even, we have the following additional cases to prove.

(5) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q_i^-(n, \frac{n}{2})$ ($i = 1, 2$) and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^I \in C(\tilde{k}, I)^+$, let $y = B \cdot G$. It is clear that $y \in \text{CTL}(D)_{m,n}^{<\tilde{k}, I)^+}$ when $\tilde{k} < \frac{n}{2}$. In the case $\tilde{k} = \frac{n}{2}$, we have

$$y \equiv \left(\prod_{i=1}^{m-1} \delta_i^{n(\bar{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, B, G)^-} \right) B'_1 \otimes G_2 \otimes C_{1,1}^\emptyset \in \text{CTL}(D)_{m,n}^{(\tilde{k}, I)^+},$$

where $B'_1 \in D^+(n, \tilde{k})$ is determined by B_1 and the coefficient of $B'_1 \otimes G_2 \otimes C_{1,1}^\emptyset$ is completely determined by the horizontal arcs in B_2 and G_1 , thus both B'_1 and the coefficient of $B'_1 \otimes G_2 \otimes C_{1,1}^\emptyset$ are independent of G_2 .

(6) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q_i^-(n, \frac{n}{2})$ ($i = 1, 2$) and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^J \in C(k, J)^-$, we always have $y = B \cdot G \in \text{CTL}(D)_{m,n}^{<(k, J)^-}$ since $k < \frac{n}{2}$.

(7) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q_1^-(n, \frac{n}{2})$ (respectively, $Q_2^-(n, \frac{n}{2})$) and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^\emptyset \in C(\frac{n}{2}, \emptyset)_1^-$ (respectively, $C(\frac{n}{2}, \emptyset)_2^-$), let $y = B \cdot G$. If the horizontal arcs in B_2 and G_1 form an interior closed cycle with one blob, then it is clear that $y \in \text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_1^-}$ (respectively, $\text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_2^-}$). Otherwise, we have

$$y \equiv \left(\prod_{i=1}^{m-1} \delta_i^{n(\bar{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, B, G)^-} \right) B_1 \otimes G_2 \otimes C_{1,1}^\emptyset \in \text{CTL}(D)_{m,n}^{(\frac{n}{2}, \emptyset)_1^-}$$

(respectively, $\text{CTL}(D)_{m,n}^{(\frac{n}{2}, \emptyset)_2^-}$), where the coefficient of $B_1 \otimes G_2 \otimes C_{1,1}^\emptyset$ is completely determined by the horizontal arcs in B_2 and G_1 , thus is independent of G_2 .

(8) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q_1^-(n, \frac{n}{2})$ (respectively, $Q_2^-(n, \frac{n}{2})$) and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^\emptyset \in C(\frac{n}{2}, \emptyset)_2^-$ (respectively, $C(\frac{n}{2}, \emptyset)_1^-$), it is always true that $y = B \cdot G \in \text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_2^-}$ (respectively, $\text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_1^-}$) since there must be an

interior closed cycle with one blob in y , which is formed by the horizontal arcs in B_2 and G_1 .

(9) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q^+(n, k)$ and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^\emptyset \in C(\frac{n}{2}, \emptyset)_1^-$ (respectively, $C(\frac{n}{2}, \emptyset)_2^-$), we always have $y = B \cdot G \in \text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_1^-}$ (respectively, $\text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_2^-}$) since there is always a closed cycle with one blob in y .

(10) For each $a = B = B_1 \otimes B_2 \otimes \omega \in Q^-(n, k)$ and each $C_{S,T}^\lambda = G = G_1 \otimes G_2 \otimes C_{1,1}^\emptyset \in C(\frac{n}{2}, \emptyset)_1^-$ (respectively, $C(\frac{n}{2}, \emptyset)_2^-$), let $y = B \cdot G$. If the horizontal arcs in B_2 and G_1 form an interior closed cycle with one blob, then it is clear that $y \in \text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_1^-}$ (respectively, $\text{CTL}(D)_{m,n}^{<(\frac{n}{2}, \emptyset)_2^-}$). Otherwise, we have

$$y = \left(\prod_{i=1}^{m-1} \delta_i^{n(\bar{i}, B, G)^+} \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, B, G)^-} \right) B' \otimes G_2 \otimes C_{1,1}^\emptyset \in \text{CTL}(D)_{m,n}^{(\frac{n}{2}, \emptyset)_1^-}$$

(respectively, $\text{CTL}(D)_{m,n}^{(\frac{n}{2}, \emptyset)_2^-}$), where B' is determined only by B_1 , B_2 and G_1 , and the coefficient of $B' \otimes G_2 \otimes C_{1,1}^\emptyset$ is completely determined by the horizontal arcs in B_2 and G_1 . Therefore, both B' and the coefficient of $B' \otimes G_2 \otimes C_{1,1}^\emptyset$ are independent of G_2 .

The above argument implies that the condition (C3) in Definition 4.1 follows. Thus the proof of the theorem is completed. \square

Remark. In particular, we obtain from the above theorem that Temperley-Lieb algebras of type D over an arbitrary field are cellular.

5 Irreducible representations and quasi-heredity of $\text{CTL}(D)_{m,n}$

The theory of cellular algebras can help us to determine all the irreducible representations of a cellular algebra. Cellular algebras and quasi-hereditary algebras are closely related. The purpose of this section is to investigate the irreducible representations of cyclotomic Temperley-Lieb algebras of type D and then determine which parameters yield a quasi-hereditary cyclotomic Temperley-Lieb algebra of type D .

We first recall that given a cellular algebra A with cell datum (Λ, M, C, i) , one can define for each $\lambda \in \Lambda$ a *cell module* $W(\lambda)$ and a bilinear form $\phi_\lambda : W(\lambda) \otimes_K W(\lambda) \rightarrow K$ as follows. As a vector space, $W(\lambda)$ has a K -basis $\{C_S^\lambda \mid S \in M(\lambda)\}$, and the module structure is given by

$$a C_S^\lambda = \sum_{U \in M(\lambda)} r_a(U, S) C_U^\lambda,$$

where the coefficients $r_a(U, S)$ are determined by (C3) in Definition 4.1. The bilinear form ϕ_λ is defined by

$$\phi_\lambda(C_S^\lambda, C_T^\lambda) C_{U,V}^\lambda \equiv C_{U,S}^\lambda C_{T,V}^\lambda$$

modulo the ideal generated by all basis elements with upper index strictly smaller than λ .

The theory of cellular algebras (see Graham and Lehrer, 1996) shows that the isomorphism classes of simple A -modules are parametrized by the set $\Lambda_0 = \{\lambda \in \Lambda \mid \phi_\lambda \neq 0\}$. It can be realized in the following way. We write $\text{rad}(\lambda)$ for the subspace of the cell module $W(\lambda)$ given by $\{x \in W(\lambda) \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in W(\lambda)\}$. It has been proven that $\text{rad}(\lambda)$ is a submodule of $W(\lambda)$. As a result, the factor module $W(\lambda)/\text{rad}(\lambda)$ with $\lambda \in \Lambda_0$ gives rise to a simple A -module. In this case, we write $S(\lambda)$ for $W(\lambda)/\text{rad}(\lambda)$.

The following theorem is a parametrization of the irreducible representations of cyclotomic Temperley-Lieb algebras of type D . To state the theorem, we introduce the following notations.

$$S_1 = \left\{ S((\tilde{k}, I)^+), S((k, J)^-) \left| \begin{array}{l} 1 \leq \tilde{k} \leq [\frac{n}{2}], 0 \leq k \leq [\frac{n}{2}], \\ I = (i_1, \dots, i_{n-2\tilde{k}}) \in \Lambda(m, n-2\tilde{k}), \\ J = (j_1, \dots, j_{n-2k}) \in \Lambda(m, n-2k) \\ \text{with all } i_h, j_h \text{ divisible by } p^t \end{array} \right. \right\}$$

$$S_2 = \left\{ S((\tilde{k}, I)^+), S((k, J)^-) \left| \begin{array}{l} 1 \leq \tilde{k} \leq [\frac{n}{2}] - 1, 0 \leq k \leq [\frac{n}{2}] - 1, \\ I = (i_1, \dots, i_{n-2\tilde{k}}) \in \Lambda(m, n-2\tilde{k}), \\ J = (j_1, \dots, j_{n-2k}) \in \Lambda(m, n-2k) \\ \text{with all } i_h, j_h \text{ divisible by } p^t \end{array} \right. \right\}$$

$$S'_2 = \left\{ S((\frac{n}{2}, \emptyset)^+), S((\frac{n}{2}, \emptyset)_1^-), S((\frac{n}{2}, \emptyset)_2^-) \right\}$$

Theorem 5.1 *Let K be a splitting field of $x^m - 1$ and $\text{Char} K = p$. Write $m = p^t s$ with $(p, s) = 1$ and $t \geq 0$ (in the case $p = 0$, set $0^0 = 1$). Then we have the following.*

(1) *If n is odd, then the set S_1 forms a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.*

(2) *If n is even.*

(i) *If not all δ_i are zero, then the set $S_2 \cup S'_2$ forms a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.*

(ii) *If all δ_i are zero, then the set S_2 is a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.*

To prove Theorem 5.1, we first recall the following lemma from Rui and Xi (2004) which describes the simple $G_{m,n}$ -modules.

Lemma 5.2 (Rui and Xi, 2004). *Let K be a splitting field of $x^m - 1$ and $\text{Char} K = p$.*

(1) *If p divides m , say, $m = p^t s$ with $(p, s) = 1$, then $\{S(I) \mid I = (i_1, \dots, i_n) \text{ with all } i_j \text{ divisible by } p^t\}$ forms a complete set of non-isomorphic simple $G_{m,n}$ -modules whose cardinality is s^n .*

(2) *If p does not divide m , then $\{S(I) \mid I \in \Lambda(m, n)\}$ is a complete set of non-isomorphic simple $G_{m,n}$ -modules. In this case, the algebra $G_{m,n}$ is semisimple.*

For any $I \in \Lambda(m, n-2k)$, let ψ_I be the bilinear form defined on the cell module of $G_{m, n-2k}$ associated with the index I . Similarly, for any $\lambda \in \Lambda_{m, n}$, let ϕ_λ be the bilinear form defined on the cell module $W(\lambda)$ of $\text{CTL}(D)_{m, n}$. To study the irreducible representations of $\text{CTL}(D)_{m, n}$, we first discuss when the bilinear form ϕ_λ is equal to zero.

Lemma 5.3 (1) If $\psi_I = 0$ for some $I \in \Lambda(m, n-2k)$ and $1 \leq k \leq [n/2]$, $k \neq \frac{n}{2}$, then $\phi_{(k, I)^+} = 0$.

(2) If $\psi_I = 0$ for some $I \in \Lambda(m, n-2k)$ and $0 \leq k \leq [n/2]$, $k \neq \frac{n}{2}$, then $\phi_{(k, I)^-} = 0$.

Proof. For any $v_1, v_2 \in D^+(n, k)$ and $I \in \Lambda(m, n-2k)$, let $y = (v_1 \otimes v_2 \otimes C_{1,1}^I) \cdot (v_1 \otimes v_2 \otimes C_{1,1}^I)$. If $y \in \text{CTL}(D)_{m, n}^{<(k, I)^+}$, then $\phi_{(k, I)^+} = 0$. Otherwise, suppose $y \equiv v_1 \otimes v_2 \otimes x C_{1,1}^I C_{1,1}^I \pmod{\text{CTL}(D)_{m, n}^{<(k, I)^+}}$, where $x \in G_{m, n-2k}$. If $\psi_I = 0$, then $C_{1,1}^I C_{1,1}^I \equiv 0 \pmod{G_{m, n-2k}^{<I}}$. Hence $x C_{1,1}^I C_{1,1}^I \equiv 0 \pmod{G_{m, n-2k}^{<I}}$ and $v_1 \otimes v_2 \otimes x C_{1,1}^I C_{1,1}^I \equiv 0 \pmod{\text{CTL}(D)_{m, n}^{<(k, I)^+}}$, thus $\phi_{(k, I)^+} = 0$. By a similar argument, we have $\phi_{(k, I)^-} = 0$ as required. \square

Lemma 5.4 Assume that $\psi_I \neq 0$ for some $I \in \Lambda(m, n-2k)$. Then $\phi_{(k, I)^+} \neq 0$ and $\phi_{(k, I)^-} \neq 0$ if n is odd or n is even but $k \neq \frac{n}{2}$.

Proof. We can show that $\phi_{(k, I)^+}$ and $\phi_{(k, I)^-}$ are not zero by choosing some special m -cyclotomic D -admissible dangles of type (n, k) . Let D_1 be the m -cyclotomic D -admissible dangle of type I (respectively, type II) with horizontal arcs $\{1 < 2\}$, $\{3 < 4\}$, \dots , $\{2k-1 < 2k\}$, and let D_2 be the m -cyclotomic D -admissible dangle of type I (respectively, type II) with horizontal arcs $\{2 < 3\}$, $\{4 < 5\}$, \dots , $\{2k < 2k+1\}$. Each horizontal arc does not carry dots and there are no blobs on each horizontal arc and vertical line. It is clear that $(D_1 \otimes D_2 \otimes C_{1,1}^I) \cdot (D_1 \otimes D_2 \otimes C_{1,1}^I) = D_1 \otimes D_2 \otimes C_{1,1}^I C_{1,1}^I$, thus we get $\phi_{(k, I)^+} \neq 0$ (respectively, $\phi_{(k, I)^-} \neq 0$) from $\psi_I \neq 0$. \square

Lemma 5.5 Assume that n is even and $k = \frac{n}{2}$. Then

(1) $\phi_{(\frac{n}{2}, \emptyset)^+} = 0$ if and only if all δ_i are zero.

(2) $\phi_{(\frac{n}{2}, \emptyset)_i^-} = 0$ ($i = 1, 2$) if and only if all δ_i are zero.

Proof. We only prove statement (2). The proof of statement (1) is similar. We again choose some special m -cyclotomic D -admissible dangles of type $(n, \frac{n}{2})$ to verify the lemma. Let D_1 be the m -cyclotomic D -admissible of type II with horizontal arcs $\{1 < n\}$, $\{2 < 3\}$, $\{4 < 5\}$, \dots , $\{n-2 < n-1\}$, and let D_2 be the m -cyclotomic D -admissible dangle of type II with horizontal arcs $\{1 < 2\}$, $\{3 < 4\}$, \dots , $\{n-1 < n\}$.

Suppose that $\delta_i \neq 0$ for some $0 \leq i \leq m-1$. Let $D_1^+ = D_1$ and D_2^+ to be the dangle decorated by i dots in the horizontal arc $\{1 < 2\}$ in D_2 . In this case, we get $(D_1^+ \otimes D_2^+ \otimes C_{1,1}^\emptyset) \cdot (D_1^+ \otimes D_2^+ \otimes C_{1,1}^\emptyset) = \delta_i (D_1^+ \otimes D_2^+ \otimes C_{1,1}^\emptyset)$. So $\phi_{(\frac{n}{2}, \emptyset)_1^-} \neq 0$. We take D_1^+ to be the dangle decorated by one blob in the horizontal arc $\{1 < n\}$ in D_1 and D_2^+ to be the dangle decorated by one blob and i dots in the horizontal arc $\{1 < 2\}$ in D_2 . In this case, we also get $(D_1^+ \otimes D_2^+ \otimes C_{1,1}^\emptyset) \cdot (D_1^+ \otimes D_2^+ \otimes C_{1,1}^\emptyset) = \delta_i (D_1^+ \otimes D_2^+ \otimes C_{1,1}^\emptyset)$, so $\phi_{(\frac{n}{2}, \emptyset)_2^-} \neq 0$.

Suppose that $\delta_i = 0$ for all $0 \leq i \leq m-1$. For any $v_1, v_2 \in D_i^-(n, \frac{n}{2})$ ($i = 1, 2$), the composition of $v_1 \otimes v_2 \otimes C_{1,1}^\emptyset$ with itself must contain at least one interior closed cycle since there are no vertical lines in the dangles v_1 and v_2 . Note that these interior closed cycles will provide some zero factors δ_i . Thereby the product $(v_1 \otimes v_2 \otimes C_{1,1}^\emptyset) \cdot (v_1 \otimes v_2 \otimes C_{1,1}^\emptyset)$ is zero, and thus $\phi_{(\frac{n}{2}, \emptyset)_i^-} = 0$ ($i = 1, 2$). \square

Now we are in the position to prove Theorem 5.1.

Proof of Theorem 5.1. Let $(\tilde{k}, I)^+ \in \Lambda_{m,n}$ and $(k, J)^- \in \Lambda_{m,n}$. In the case $(\tilde{k}, I)^+ = (\frac{n}{2}, \emptyset)^+$ (respectively, $(k, J)^- = (\frac{n}{2}, \emptyset)_i^-$ ($i = 1, 2$)), we know by Lemma 5.5 that $\phi_{(\frac{n}{2}, \emptyset)^+} = 0$ (respectively, $\phi_{(\frac{n}{2}, \emptyset)_i^-} = 0$ ($i = 1, 2$)) if and only if $\delta_i = 0$ for all $0 \leq i \leq m-1$. If $(\tilde{k}, I)^+ \neq (\frac{n}{2}, \emptyset)^+$ (respectively, $(k, J)^- \neq (\frac{n}{2}, \emptyset)_i^-$ ($i = 1, 2$)), then by Lemma 5.3 and Lemma 5.4 we see that $\phi_{(\tilde{k}, I)^+} = 0$ (respectively, $\phi_{(k, J)^-} = 0$) if and only if $\psi_I = 0$ (respectively, $\psi_J = 0$). Moreover, by Lemma 5.2 we know that $\psi_I \neq 0$ if and only if p^t divides all i_h , where i_h are the components of $I = (i_1, i_2, \dots, i_{n-2\tilde{k}}) \in \Lambda(m, n-2\tilde{k})$ and $\psi_J \neq 0$ if and only if p^t divides all j_h , where j_h are the components of $J = (j_1, j_2, \dots, j_{n-2k}) \in \Lambda(m, n-2k)$. This finishes the proof. \square

In the rest of this section we will determine which parameters yield a quasi-hereditary cyclotomic Temperley-Lieb algebra of type D . Quasi-hereditary algebras are used to describe the highest weight categories appearing in the representation theory of semisimple Lie algebras and algebraic groups. We first recall the definition of quasi-hereditary algebras.

Definition 5.6 (Cline, Parshall and Scott, 1988). Let A be a K -algebra. An ideal J of A is called a *heredity ideal* if J is idempotent, $J(\text{rad} A)J = 0$, and J is a projective left (or, right) A -module. The algebra A is called *quasi-hereditary* provided there is a finite chain $0 = J_0 \subset J_1 \subset \dots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal of A/J_{j-1} for all j . Such a chain is then called a *heredity chain* of the quasi-hereditary algebra A .

As indicated by the ideals appearing in cell chains, there are close connections between cellular algebras and quasi-hereditary algebras. The class of cellular algebras has a large intersection with the class of quasi-hereditary algebras. Typical examples of quasi-hereditary algebras obtained from cellular algebras include Temperley-Lieb algebras of type A with non-zero parameters (see Westbury, 1995) and Birman-Wenzl algebras for most choices of parameters (see Xi, 2000). Recently, Xi (2002) proved that a cellular algebra is quasi-hereditary if and only if the first cohomology groups between cell modules and dual modules are always trivial. A stronger statement describing the quasi-heredity of cellular algebras was given later by Cao (2003). One can refer to the survey paper by König and Xi (1999c) for a comparison between cellular algebras and quasi-hereditary algebras. Now we recall the following theorem which determines those cellular algebras which are quasi-hereditary.

Theorem 5.7 (König and Xi, 1999a). *Let K be a field and A a cellular K -algebra (with respect to an involution i). Then the following statements are equivalent.*

(1) Some cell chain of A (with respect to some involution, possibly different from i) is a heredity chain as well, thus it makes A into a quasi-hereditary algebra.

(1') There is a cell chain of A (with respect to some involution, possibly different from i) whose length equals the number of isomorphism classes of simple A -modules.

(2) A has finite global dimension.

(3) The Cartan matrix of A has determinant one.

(4) Any cell chain of A (with respect to any involution) is a heredity chain.

As an immediate consequence of the above theorem, we have the following corollary.

Corollary 5.8 *Let K be a field and A a cellular K -algebra. Then A is quasi-hereditary if and only if $|\Lambda| = |\Lambda_0|$.*

Proof. Suppose that $|\Lambda| \neq |\Lambda_0|$. Then there exists a cell chain which is not a heredity chain since there exists a cell ideal which is not a heredity one. So A is not quasi-hereditary from the statement (4) in Theorem 5.7.

If $|\Lambda| = |\Lambda_0|$, then there exists a cell chain whose length equals the number of isomorphism classes of simple A -modules. So A is quasi-hereditary from the statement (1') in Theorem 5.7. \square

The following theorem gives a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebras of type D to be quasi-hereditary.

Theorem 5.9 *Let K be a splitting field of $x^m - 1$ and let $\text{Char} K = p$. Then the cyclotomic Temperley-Lieb algebra of type D is quasi-hereditary if and only if p does not divide m and one of the following holds:*

(1) n is odd.

(2) n is even, but not all δ_i are zero.

Proof. By Corollary 5.8, we know that $\text{CTL}(D)_{m,n}$ is quasi-hereditary if and only if the index set of cell modules of $\text{CTL}(D)_{m,n}$ coincides with that of simple modules. Moreover, the coincidence occurs if and only if p does not divide m , and either n is odd or not all δ_i are zero by Theorem 5.1. \square

For the case not displayed in Theorem 5.9, we may get a quasi-hereditary quotient of $\text{CTL}(D)_{m,n}$.

Proposition 5.10 *Suppose that K is a splitting field of $x^m - 1$ and that p does not divide m . If n is even and all δ_i are zero, then the factor algebra $\text{CTL}(D)_{m,n}/J$ is quasi-hereditary, where J is the ideal of $\text{CTL}(D)_{m,n}$ generated by all m -cyclotomic D -admissible diagrams without vertical arcs.*

Proof. If n is even and all δ_i are zero, then the ideal J generated by all m -cyclotomic D -admissible diagrams without vertical arcs is nilpotent by Lemma 5.5. It is clear that the factor algebra $\text{CTL}(D)_{m,n}/J$ is again cellular with respect to the induced involution. In addition, under the assumption of the proposition the index set of cell modules of $\text{CTL}(D)_{m,n}/J$ coincides with that of simple $\text{CTL}(D)_{m,n}/J$ -modules. Therefore $\text{CTL}(D)_{m,n}/J$ is a quasi-hereditary algebra. \square

6 An example

In this section, we will go through a concrete example of cyclotomic Temperley-Lieb algebras of type D in the case $m = 2$ and $n = 4$ to illustrate the results in the previous sections.

In the case $n = 4$, the K -dimension of Temperley-Lieb algebra of type D is 48 with 13 D -admissible diagrams of type I and 35 D -admissible diagrams of type II. In the case $m = 2$ and $n = 4$, the number of 2-cyclotomic D -admissible diagrams of type I is 208 and the number of m -cyclotomic D -admissible diagrams of type II is 560. This is a total of 768 which is the K -dimension of $\text{CTL}(D)_{2,4}$.

We first describe the cellular structure of $\text{CTL}(D)_{2,4}$. Recall that $\Lambda(2, 4) = \{(j_1, j_2, j_3, j_4) \mid 1 \leq j_h \leq 2\}$ and $\Lambda(2, 2) = \{(i_1, i_2) \mid 1 \leq i_h \leq 2\}$. The partial orderings of $(\Lambda(2, 4), \preceq)$ and $(\Lambda(2, 2), \preceq)$ are illustrated by the following figures. Two elements connected in the figures are comparable and the larger element sits above the smaller.

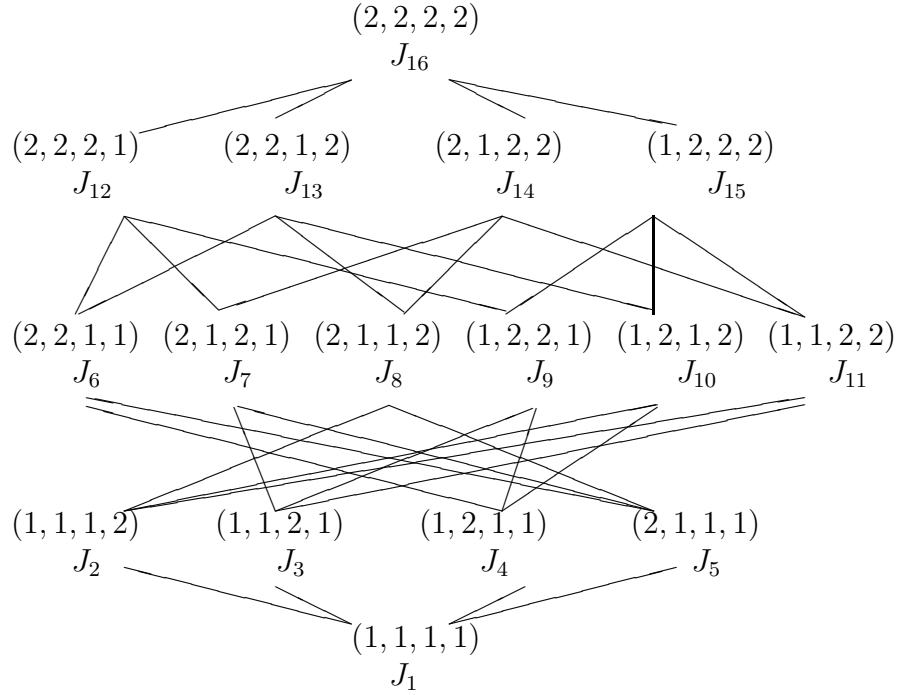


Figure 6. Partially ordered set $(\Lambda(2, 4), \preceq)$

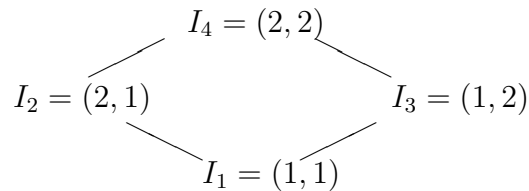


Figure 7. Partially ordered set $(\Lambda(2, 2), \preceq)$

The partial ordering of $(\Lambda_{2,4}, \leq)$ can be illustrated by the following figure.

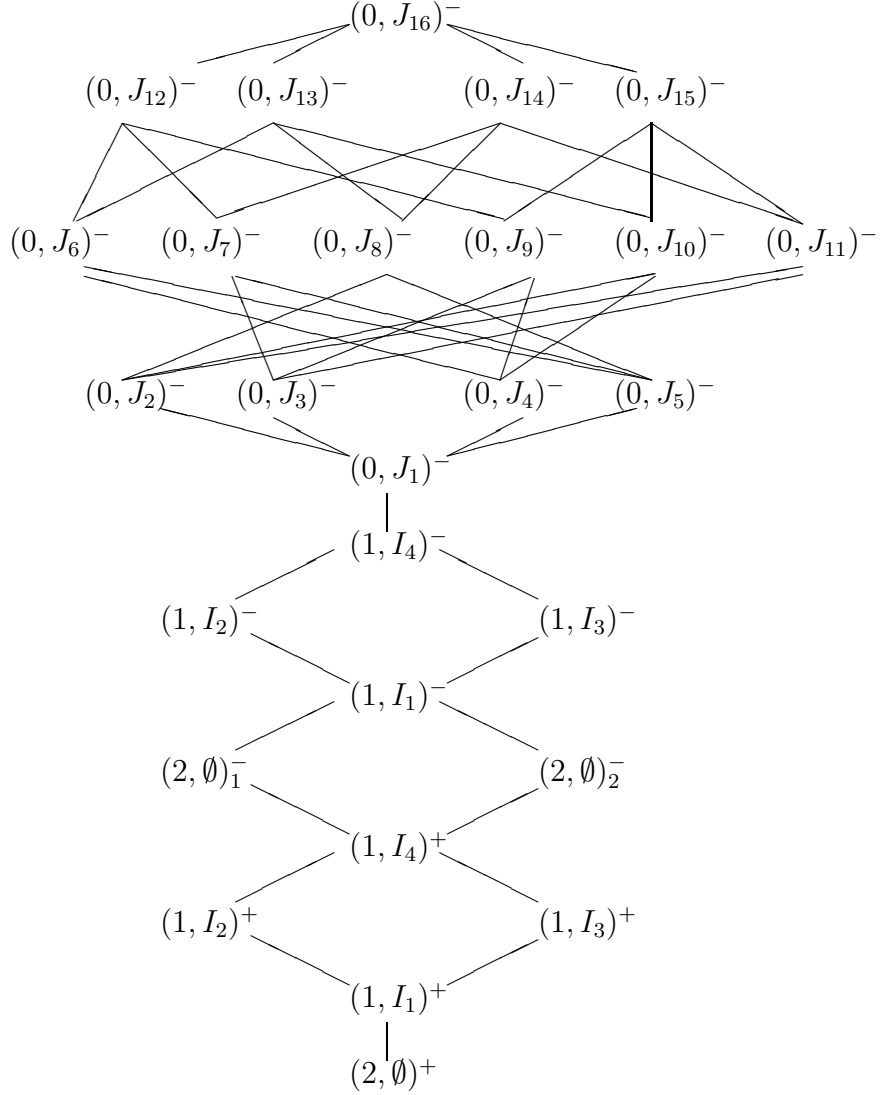


Figure 8. Partially ordered set $(\Lambda_{2,4}, \leq)$

Suppose $v_1, v_2 \in D^-(4, 1)$ are 2-decorated dangles as follows.

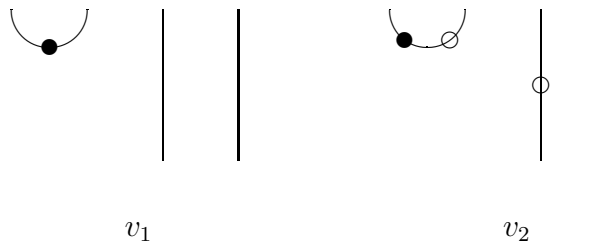


Figure 9. 2-Decorated dangles

For $(1, I_2)^- \in \Lambda_{2,4}$ and $(v_1, I_2), (v_2, I_2) \in M((1, I_2)^-)$, we have $C_{(v_1, I_2), (v_2, I_2)}^{(1, I_2)^-} = v_1 \otimes v_2 \otimes C_{1,1}^{I_2}$ as follows.

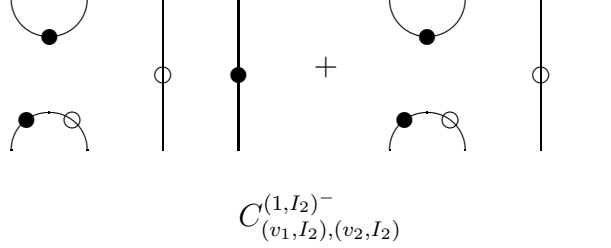


Figure 10. An element of the cellular basis

We can write all other elements of the cellular basis of $\text{CTL}(D)_{2,4}$ in a similar way.

Secondly, by Theorem 5.1 we can determine all the irreducible representations of $\text{CTL}(D)_{2,4}$. There are four cases.

(1) If $\text{Char}K = 0$ or $\text{Char}K = p \geq 3$ and $\delta_0 \neq 0, \delta_1 = 0$ or $\delta_0 = 1, \delta_1 \neq 0$, then the set $\{S(\lambda) \mid \lambda \in \Lambda_{2,4}\}$ forms a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.

(2) If $\text{Char}K = 0$ or $\text{Char}K = p \geq 3$ and $\delta_0 = \delta_1 = 0$, then the set $\{S(\lambda) \mid \lambda \in \Lambda_{2,4} \setminus \{S((2, \emptyset)^+), S((2, \emptyset)_1^-), S((2, \emptyset)_2^-)\}\}$ forms a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.

(3) If $\text{Char}K = 2$ and $\delta_0 \neq 0, \delta_1 = 0$ or $\delta_0 = 1, \delta_1 \neq 0$, then the set $\{S((1, I_4)^+), S((1, I_4)^-), S((0, J_{16})^-), S((2, \emptyset)^+), S((2, \emptyset)_1^-), S((2, \emptyset)_2^-)\}$ forms a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.

(4) If $\text{Char}K = 2$ and $\delta_0 = \delta_1 = 0$, then the set $\{S((1, I_4)^+), S((1, I_4)^-), S((0, J_{16})^-)\}$ forms a complete set of non-isomorphic simple $\text{CTL}(D)_{m,n}$ -modules.

Finally, we know by Theorem 5.9 that $\text{CTL}(D)_{2,4}$ is quasi-hereditary if and only if $\text{Char}K = 0$ or $\text{Char}K = p \geq 3$ and $\delta_0 \neq 0, \delta_1 = 0$ or $\delta_0 = 1, \delta_1 \neq 0$.

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